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Maximal arc partitions of designs

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Abstract

It is known that the designs $PG_{n-1}(n,q)$ in some cases have spreads of maximal α-arcs. Here a α-arc is a non-empty subset of points that meets every hyperplane in 0 or α points. The situation for designs in general is not so well known. This paper establishes an equivalence between the existence of a spread of α-arcs in the complement of a Hadamard design and the existence of an affine design and a symmetric design which is also the complement of a Hadamard design.

MSC: 51E05

Keywords: Design; Hadamard 2-design; Arc

1. Introduction

An α-arc in a 2-design is a subset of points that meets every block in either 0 or α points. [7,8].

Rahilly [6] established the equivalence of the existence of an affine design of class number 4 and a Hadamard 2-design possessing a spread of lines of maximum size 3. By observing that a line of maximum size 3 in a Hadamard design is a 1-arc in the complementary design, we are able to extend this result and to state it in the language of maximal arcs in designs.

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الأوراق العلمية المقدمة للترقية

1. A. N. Al-Kenani, N. J. Khojah, and S. K. Nauman, g-homomorphisms and Morphisms between Morita Contexts, The *Alig. Bulletin of Math.*, 23(2004),33-48.
2. A. N. Al-Kenani and V.C. Mavron, Maximal Arc partitions of Designs, *Discrete Math.*, 294(2005), 5-11.
3. A. N. Al-Kenani, Commutativity of Rings with Polynomial Constraints, Int. J. of Pure and App. Maths, Vol. 41, No. 8, 2007, 1135-1142.
4. A. N. Al-Kenani, Algebraic presentation of Classical Nets, Advances and Applications in Discrete Mathematics, Volume 1 No. 2 (2008), 187 - 194.
5. A. N. Al-Kenani and S. K. Nauman, A short construction of MSM rings, JP, Journal of Algebra, Number Theory and Applications, Vol. 11, No. 2(2008),203-207.
6. A. N. Al-Kenani, Polynomial Constraints on Some Rings, Int. J. of Pure and App. Maths, Accepted.

affine case can be expressed entirely in terms of μ and m as follows: $v = \mu m^2$, $k = \mu m$, $\lambda = (\mu m - 1)/(m - 1)$, $r = (\mu m^2 - 1)/(m - 1)$ and $b = rm$.

A 2-(v, k, λ) design D is *symmetric* if $b = v$. It is well-known that D is symmetric if and only if its dual design D^* is also a 2-(v, k, λ) design.

A *Hadamard 2-design* is a symmetric 2-(v, k, λ) design with $v = 4\lambda + 3$ and $k = 2\lambda + 1$. Such a design exists if and only if there exists a Hadamard matrix of order $v + 1$. A *complementary Hadamard* 2-design is the complement of a Hadamard 2-design; so its parameters are of the form 2-$(4\lambda + 3, 2\lambda + 2, \lambda + 1)$. The Hadamard conjecture asserts that a Hadamard matrix of order n exists if and only if $n = 2$ or n is divisible by 4.

Given a Hadamard 2-$(4\lambda + 3, 2\lambda + 1, \lambda)$ design D, introduce a new point w and adjoin it to each block of D. These extended blocks and their complements give an affine 3-$(4\lambda + 4, 2\lambda + 2, \lambda)$ design. Any affine 2-design of class number 2 is in fact a 3-design obtained in this way from some (not necessarily unique) Hadamard 2-design.

The preceding discussion relating Hadamard matrices to particular classes of symmetric designs and affine designs of class number 2 is well-known. The idea has roots in a paper of Bose [2]. However, Rahilly [6] showed that there is a connection between Hadamard 2-designs and affine designs of class number 4.

Proposition 2 (*Rahilly [6]*). *There exists an affine* 2-$(16\mu, 4\mu, (4\mu - 1)/3)$ *design if and only if there exists a Hadamard* 2-$(16\mu - 1, 8\mu - 1, 4\mu - 1)$ *design with a spread of lines, all of maximum size* 3.

In this paper, we shall extend Rahilly's result to affine designs of class number m, where $m \geqslant 4$. To this end we extend the concept of lines of maximum size. One might think that this means considering, for example, plane spreads but it turns out that considering spreads of α-arcs in complements of Hadamard 2-designs leads more naturally to a generalization of Rahilly's theorem.

Rahilly's results on line spreads were for symmetric designs. We shall consider the more general theory of spreads of α-arcs in the wider setting of 2-designs, which need not be symmetric.

3. Spreads and α-arcs

First in this section, it will be shown that a line in a design D may be viewed as an α-arc in the complementary design $\overline{D}$.

Lemma 3. *Let D be a* 2-(v, k, λ) *design $k \geqslant 3$. Then a subset of points of D is a maximum line in $\overline{D}$ if and only if it is an α-arc in D with $\alpha = r/(r - \lambda)$.*

Proof. Let A be an α-arc in D, where $\alpha = r/(r - \lambda)$. By definition, $|A| = 1 + r(\alpha - 1)/\lambda = 1 + r/(r - \lambda)$. Therefore $|A| \geqslant 2$ and so any block of D meets A in 0 or $r/(r - \lambda)$ points; hence any block of $\overline{D}$ either contains A or meets A in exactly one point. Each of the blocks that contains two distinct points of A must therefore contain all of A and hence the line joining the two points. From the previous section, we know that a maximum line of $\overline{D}$ has exactly $1+(v-1)/(v-k)$ points, which is easily shown to equal $|A|$ using the basic design parameter relations. □

Hence A is a line in $\overline{D}$. The converse is straightforward.

If A is an α-arc of D, then D_A denotes the *induced design* defined on the points of A, whose blocks are the secants of A, with induced incidence. Thus a secant B *induces* a block of D_A whose points are those of $A \cap B$. Clearly D_A is a 1-(a, α, r) design, where $|A| = a$ and r is the replication number of D. The following lemma is essentially in [8] but we include the proof for completeness.

Lemma 4. *Let A be an α-arc in a 2-(v, k, λ) design D. Then*

(a) *D_A is a 2-(a, α, λ) design, where $a = |A| = 1 + r(\alpha - 1)/\lambda$,*
(b) *A has exactly ra/α secants and $b - ra/\alpha$ passants,*
(c) *any point not in A is on exactly $\lambda a/\alpha$ secants and $r - \lambda a/\alpha$ passants,*
(d) *the passants of A form an $(r - \lambda)/\alpha$-arc in D^*.*

Proof. Condition (a) is straightforward. Moreover, for D_A the parameters 'r' and 'b' are, respectively, the replication number r of D and the number of secants of A. The standard equation '$bk = vr$' then gives (b).

To prove (c) let p be a point not in A and N the number of secants on p. Counting in two ways the number of flags (q, B), where B is a secant on p and $q \in A \cap B$, gives $a\lambda = N\alpha$. Finally, (d) follows easily from (c). □

Next, we consider the number of common secants and passants of two disjoint arcs.

Lemma 5. *Let A_i be an α_i-arc and $|A_i| = a_i$ for $i = 1, 2$, where $A_1 \cap A_2 = \emptyset$. Then the number of secants common to A_1 and A_2 is $\lambda a_1 a_2/\alpha_1\alpha_2$ and the number of common passants is $b - (a_1\alpha_2 + a_2\alpha_1 - \lambda a_1 a_2)/\alpha_1\alpha_2$.*

Proof. Let x be the number of common secants. Counting in two ways the number of ordered triples (p_1, p_2, B), where $p_i \in A_i$ and B is a block containing p_i $(i = 1, 2)$, gives $a_1 a_2 \lambda = x\alpha_1\alpha_2$. The rest is straightforward using this result and Lemma 4. □

Remark 6. Rahilly [6] defines a spread of maximum lines to be *uniform* if the number of blocks containing any two lines of the spread is constant. He then proves that every spread of maximum lines in a Hadamard 2-design is uniform. However, this is true for all 2-designs as can easily be deduced from Lemmas 3 and 5.

The *mth multiple* design of a design is obtained by repeating each of its blocks m times.

The case when the induced design on an α-*arc* is a multiple of a symmetric design is of special interest. Let D be a 2-(v, k, λ) design with an α-*arc* A. Then D_A is a 2-(a, α, λ) design, where $a = 1 + r(\alpha - 1)/\lambda$ and the replication number of D_A is r, that of D. Hence if D_A is a multiple of a symmetric design, then it is the (r/α)th multiple of a symmetric 2-(a, α, λ') design denoted by $[D_A]$, where $\lambda' = \lambda\alpha/r$. In this case we shall say that A is a *symmetric* α-*arc*.

A set of α-*arcs* that partitions the point set of D will be called an α-*spread*. If all the α-*arcs* in the spread are symmetric, it is called a *symmetric* α-*spread*.

In view of Lemma 3, every $r/(r-\lambda)$-spread in D is a line spread in $\overline{D}$ in the sense of Rahilly [6]: that is a partition of the point set by maximum lines. We shall show that in the case $\alpha = r/(r-\lambda)$, all α-arcs and α-spreads are symmetric.

Lemma 7. *Every* $[r/(r-\lambda)]$*-arc in a* 2-(v, k, λ) *design is symmetric and is a maximum line in the complementary design.*

Proof. First note that if x is a point of a maximum line of a 2-(v, k, λ) design, the number of blocks containing x but not the whole line is $r-\lambda$, the order of the design.

Now suppose A is an α-arc of a 2-(v, k, λ) design D, where $\alpha = r/(r-\lambda)$. Then $|A| = 1+\alpha$ and D_A is a 2-$(\alpha+1, \alpha, \alpha-1)$ design. By Lemma 3, A is a maximum line in $\overline{D}$. Therefore, given a point of A, the number of blocks of $\overline{D}$ meeting A only at that point is the order of $\overline{D}$, which is the same as the order $r-\lambda = r/\alpha$ of D. Hence each block of D_A is repeated r/α times and so A is a symmetric α-arc. □

Theorem. *There exists an affine* 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ *design and a complementary Hadamard* 2-$(m-1, \frac{1}{2}m, \frac{1}{4}m)$ *design if and only if there exists a complementary Hadamard* 2-$(\mu m^2-1, \frac{1}{2}\mu m^2, \frac{1}{4}\mu m^2)$ *design with a symmetric* $\frac{1}{2}m$*-spread.*

Proof. First assume there exists an affine 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ design Γ and a 2-$(m-1, \frac{1}{2}m, \frac{1}{4}m)$ design Δ.

Choose a point w of Γ. Then on the remaining $\mu m^2 - 1$ points of Γ define a design Π whose blocks are obtained thus. For each parallel class C of Γ, identify the $m-1$ blocks of C not on w with the points of Δ. Then the union of the $\frac{1}{2}m$ blocks of Γ corresponding to a block of Δ is defined to be a block of Π.

Hence Π has $\mu m^2 - 1$ points and $\mu m \times \frac{1}{2}m = \frac{1}{2}\mu m^2$ points on each block. To evaluate the replication number of Π, let x be any of its points. There are '$r-\lambda$' $= \mu m$ parallel classes of C of Γ such that x and w are on different blocks from C.

The block of C on x, considered as a point of Δ, is in $\frac{1}{2}m$ blocks of Π. Hence x is on $\frac{1}{2}m$ blocks of Π induced by C. Therefore, in total, x is on $(\frac{1}{2}m) \times (\mu m) = \frac{1}{2}\mu m^2$ blocks of Π. It follows that Π is a symmetric design since '$r = k$'.

Now consider two distinct blocks X and Y of Π. If they are induced by the same parallel class C of Γ, then from the parameters of Δ it follows that X and Y have $\frac{1}{2}m$ blocks of C in common and therefore meet in $(\frac{1}{4}m) \times (\mu m) = \frac{1}{4}\mu m^2$ points of Π.

Suppose on the other hand, that X and Y are induced by different parallel classes of Γ. Since X and Y each consists of $\frac{1}{2}m$ blocks of Γ and non-parallel blocks of Γ meet in μ points, it follows that X and Y meet in exactly $\mu \times (\frac{1}{2}m)^2 = \frac{1}{4}\mu m^2$ points of Π.

Hence the dual of Π is a symmetric 2-design. Therefore Π and its dual Π^* are symmetric 2-designs with parameters 2-$(\mu m^2 - 1, \frac{1}{2}\mu m, \frac{1}{4}\mu m)$.

Next, we show that Π^* has a symmetric $\frac{1}{2}m$ spread. Let C be any parallel class of Γ and x any point of Π. Let X be the block of C on x. If also w is on X, then no block of Π induced by C contains x. Otherwise the number of blocks on x induced by C is the number of blocks containing X (considered as a point of Δ) which is therefore the replication number $\frac{1}{2}m$ of Δ. Hence the $m-1$ blocks of Π induced by C form an α-arc in Π^*, where $\alpha=\frac{1}{2}m$. We show this arc is symmetric, noting here that $r/\alpha=\frac{1}{2}\mu m^2/\frac{1}{2}m=\mu m$.

In the case when x is on $\frac{1}{2}m$ blocks of Π (induced by C), all the μm points of X are on the same $\frac{1}{2}m$ blocks. This shows that the $m-1$ blocks induced by C form a symmetric $\frac{1}{2}m$-arc in Π^*.

Clearly, by varying C over all parallel classes of Γ, we obtain a symmetric $\frac{1}{2}m$-spread in Π^*.

Conversely, assume the existence of a 2-$(\mu m^2-1, \frac{1}{2}\mu m^2, \frac{1}{4}\mu m^2)$ design D with a symmetric $\frac{1}{2}m$-spread Σ. Let $A\in\Sigma$. Then A is a symmetric $\frac{1}{2}m$-arc. Further, by Lemma 4, $|A|=m-1, A$ has $\mu m(m-1)$ secants and $\mu m-1$ passants. Since A is a symmetric $\frac{1}{2}m$-*arc* it follows easily that D_A is a symmetric 2-$(m-1, \frac{1}{2}m, \frac{1}{4}m)$ design.

Define a design Γ as follows. The points of Γ are those of D^* and a new point, labelled w. The blocks of Γ are of two types. Type 1 blocks are labelled $\langle A\rangle$, $A\in\Sigma$. Hence there are $(\mu m^2-1)/(m-1)$ blocks of Type 1.

Type 2 blocks of Γ are labelled $\langle A, e\rangle$, where $A\in\Sigma$ and e is any block of $[D_A]$. Hence since $|\Sigma|=(\mu m^2-1)/(m-1)$ and each $[D_A]$ has $m-1$ blocks, it follows that there are μm^2-1 blocks of Type 2. Therefore Γ has exactly $m(\mu m^2-1)/(m-1)$ blocks.

Finally to complete the definition of Γ, we define incidence in Γ.

(i) If $A\in\Sigma$, then $\langle A\rangle$ is incident with w and with all the passants of A in D: they are points of D^* and therefore of Γ. By Lemma 4, $\langle A\rangle$ is on exactly $1+(\mu m-1)=\mu m$ points.

(ii) Let $\langle A, e\rangle$ be a Type 2 block as defined above. Each block e of $[D_A]$ is the intersection with A of any one of μm secants of A in D, since A is symmetric; so that each block of D_A is repeated 'r/α' times. (Here $r=\frac{1}{2}\mu m^2$ and $\alpha=\frac{1}{2}m$.) These μm secants as points of D^* are defined to be incident with $\langle A, e\rangle$ in Γ.

Hence Γ has μm^2 points, with μm points on each block. Next, we show Γ is a 2-design. Consider two distinct points X and Y of Γ. There are two cases.

Case 1: $Y=w$. Then only Type 1 blocks contain X and Y and the number of such blocks is the number ρ of $A\in\Sigma$ for which Y is a passant in D. Since Σ partitions the points of D and Y is a secant to $(\mu m^2-1)/(m-1)-\rho$ of the $\frac{1}{2}m$-arcs in Σ, then $(\mu m^2-1)/(m-1)-\rho=(\frac{1}{2}\mu m^2)/(\frac{1}{2}m)=\mu m$, whence $\rho=(\mu m-1)/(m-1)$.

Case 2: Neither X nor Y is w. Let π be the number of $A\in\Sigma$ such that X and Y are both passants of A in D. Then exactly $\sigma=(\mu m^2-1)/(m-1)-2\rho+\pi$ of the arcs $A\in\Sigma$ are such that X and Y are both secants of A. Furthermore, π is the number of Type 1 blocks of Γ containing both X and Y.

Let τ be the number of Type 2 blocks of Γ containing X and Y. We need to evaluate $\pi+\tau$. First observe that X and Y are both secants to exactly σ of the arcs in Σ. That is they induce the same block in τ of the symmetric 2-$(m-1, \frac{1}{2}m, \frac{1}{4}m)$ designs $[D_A]$, and induce different blocks in the $\sigma-\tau$ remaining $[D_A]$, where $A\in\Sigma$ and X, Y are both secants of A. That is, for τ of the arcs $A\in\Sigma$, the blocks $A\cap X$ and $A\cap Y$ of $[D_A]$ are equal, so that $|A\cap X\cap Y|=|A\cap X|=\frac{1}{2}m$; while for $\sigma-\tau$ of the arcs, $A\cap X$ and $A\cap Y$ meet in $\frac{1}{4}m$ points, so that $|A\cap X\cap Y|=\frac{1}{4}m$. For the remaining $A\in\Sigma$, either X or Y is a passant, so that $A\cap X\cap Y=\phi$.

Since from the parameters of the symmetric design D we have $|X \cap Y| = \frac{1}{4}\mu m^2$, it follows that $\frac{1}{4}\mu m^2 = \frac{1}{2}m\tau + \frac{1}{4}m(\sigma - \tau)$, whence $\mu m = \sigma + \tau$. Substituting for σ and ρ we obtain $\pi + \tau = (\mu m - 1)/(m - 1) = \rho$.

It follows that Γ is a 2-$(\mu m^2, \mu m, (\mu m - 1)/(m - 1))$ design. A straightforward check will verify that Γ is resolvable: a typical parallel class is given by each $A \in \Sigma$ and consists of the block $\langle A \rangle$ together with the $m - 1$ blocks $\langle A, e \rangle$, where e is any of the $m - 1$ blocks of $[D_A]$. Hence from Bose's theorem (see Section 1) it follows that Γ is affine. □

As a corollary we can readily obtain the proposition due to Rahilly [6] stated earlier. Since a 2-(3, 2, 1) design always exists, then for $m = 4$ the above theorem states that the existence of an affine 2-$(16\mu, 4\mu, \frac{1}{3}(4\mu - 1))$ design is equivalent to the existence of a complementary Hadamard 2-$(16\mu - 1, 8\mu, 4\mu)$ design with a symmetric 2-spread. Now apply Lemma 3.

An interesting case is $m = 4$, $\mu = 7$. Then the theorem implies that the existence of an affine 2-(112, 28, 9) design is equivalent to the existence of a Hadamard 2-(111, 55, 27) design with a spread of lines, all of size 3. The existence of such an affine design is undecided. According to Tonchev, it is the smallest undecided affine 2-design: on the other hand, there exist Hadamard designs on 111 points but it is not known whether any of them have spreads.

Examples of spreads of α-arcs are to be found in the designs $\mathrm{PG}_{n-1}(n, q)$ of the points and hyperplanes in $\mathrm{PG}(n, q)$. If $t + 1$ divides $n + 1$, then $\mathrm{PG}_{n-1}(n, q)$ contains a spread of t-dimensional subspaces which in the complementary design is a symmetric q^t-spread. See, e.g. [3].

Jungnickel and Tonchev [5] showed that there exist symmetric designs with the parameters of, but not isomorphic to $\mathrm{PG}_{n-1}(n, q)$, namely GMW designs, possessing spreads of α-arcs.

References

[1] T. Beth, D. Jungnickel, H. Lenz, Design Theory, second ed., Cambridge University Press, Cambridge, 1999.
[2] R.C. Bose, On a resolvable series of balanced incomplete block designs, Sankhyā 8 (1947) 249–256.
[3] C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, Florida, USA, 1996.
[4] D.R. Hughes, F.C. Piper, Design Theory, Cambridge University Press, Cambridge, 1985.
[5] D. Jungnickel, V.D. Tonchev, Decompositions of difference sets, J. Algebra 217 (1999) 21–39.
[6] A. Rahilly, On the line structure of designs, Discrete Math. 92 (1991) 291–303.
[7] S.S. Sane, S.S. Shrikhande, N.M. Singhi, Maximal arcs in designs, Graphs Combin. 1 (1985) 97–106.
[8] T. van Trung, Maximal arcs and related designs, J. Combin. Theory Ser. A 57 (1991) 294–301.

$$
\begin{array}{ccc}
\sum(n_i \otimes m_1) & \longrightarrow & \sum\langle n_i, m_i\rangle_{A_1} \\
\downarrow & & \downarrow \\
\sum[\nu_{12}(n_i) \otimes \mu_{12}(m_i)] & \longrightarrow & \sum[\nu_{12}(n_i)\mu_{12}(m_i)] \\
\downarrow & & \downarrow \\
\sum[\nu_{23}(n_i) \otimes \mu_{23}(m_i)] & \longrightarrow & \sum[\nu_{23}(n_i)\mu_{23}(m_i)]
\end{array}
$$

Similarly, by interchanging the variables, the other diagram can also be considered. Hence

Proposition 3.1.1. If $\kappa_{ij} = \langle \alpha_{ij}, \mu_{ij}, \nu_{ij}, \beta_{ij}\rangle : K_i \to K_j$ are MC morphisms, then the composition $\kappa_{jk} \circ \kappa_{ij} : K_1 - rightarrow K_2$ is also an MC morphism.

Examples 3.1.2. Let $K = (A, M, N, B)$ be an MC and let M_1 and N_1 be submodules of M and N, respectively. If $K_1 = (A, M_1, N_1, B)$ is also an MC, then $\kappa = \langle 1_A, \mu, \nu, 1_B\rangle : K_1 \to K$ is a morphism of MCs K_1 into K, where μ and ν are the embeddings $\mu = i_{M_1} : M_1 \to M$ and $\nu = i_{N_1} : M_1 \to M$. In [5], Müller called K_1 a subcontext of K. If we assume $\bar{K} = (A, M/M_1, N/N_1, B)$ and $\bar{K}$ is also an MC, then $\kappa = \langle 1_A, \mu, \nu, 1_B\rangle : K \to \bar{K}$ is an MC morphism, where μ and ν are the natural epimorphisms. $\bar{K}$ is a homomorphic image of K.

Following example is a continuation of Example 2.1.1.

Example 3.1.3. Let $B_1 = R$ be any ring and $A_1 = M_n(R)$, $M_1 = R^{(n)}$ (row wise), and $N_1 = {}^{(n)}R$ (column wise). Considering M_1 a (B_1, A_1) - bimodule and N_1 an (A_1, B_1) - bimodule, one can always get an MC, $K_1 = (A_1, M_1, N_1, B_1)$ where the first MC map $\langle,\rangle_{A_1}$ is defined by the dyads

$$
\langle \begin{bmatrix} n_1 \\ \vdots \\ n_n \end{bmatrix}, [m_1 \cdots m_n]\rangle_{A_1} = \begin{bmatrix} n_1m_1 & \cdots & n_1m_n \\ \vdots & \cdots & \vdots \\ n_nm_1 & \cdots & n_nm_n \end{bmatrix} \in A_1
$$

and the second MC map $\langle,\rangle_{B_1}$ is defined by the dot product

$$
\langle [m_1 \cdots m_n], \begin{bmatrix} n_1 \\ \vdots \\ n_n \end{bmatrix}\rangle_{B_1} = m_1n_1 + \cdots + m_nn_n \in B_1
$$

If we choose another ring, say, $B_2 = S$, then on the similar pattern one can construct another MC $K_2 = (A_2, M_2, N_2, B_2)$.

Let $f : R \to S$ be a homomorphism of rings. Then

$$
\kappa = \langle f_{(n)}, \mu, \nu, f\rangle
$$

is a morphism of MCs from K_1 into K_2, where $f_{(n)} : A_1 \to A_2$ and $\mu : M_1 \to M_2$ are as defined in Example 2.2.1 and $\nu : N_1 \to N_2$ can similarly be defined as μ, but on column vectors. Clearly, $\kappa = \langle f_{(n)}, \mu, \nu, f\rangle$ mostly depends on $f : B_1 \to B_2$. In particular, if f is monic or epic then so is κ.

3.2. Morphisms Between Rings of Morita Contexts

For any MC $K_i = (A_i, M_i, N_i, B_i)$, let us denote its context ring by $T_i = \begin{bmatrix} A_i & N_i \\ M_i & B_i \end{bmatrix}$. Define map

$$\tau = \begin{bmatrix} \alpha & \nu \\ \mu & \beta \end{bmatrix} : T_1 \to T_2$$

by

$$\begin{bmatrix} \alpha & \nu \\ \mu & \beta \end{bmatrix} \begin{bmatrix} a & n \\ m & b \end{bmatrix} = \begin{bmatrix} \alpha(a) & \nu(n) \\ \mu(m) & \beta(b) \end{bmatrix}$$

Then we have

Examples 3.2.1. Let $K_i = (A_i, M_i, N_i, B_i)$ be MCs and $\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K_1 \to K_2$ an MC morphism. Let T_i be the MC rings of K_i. Then the map $\tau = \begin{bmatrix} \alpha & \nu \\ \mu & \beta \end{bmatrix} : T_1 \to T_2$ is an identity preserving ring homomorphism. Moreover, $\mathrm{Ker}(\tau)$ is an ideal of T_1 and if μ is (β, α) - strong and ν is (α, β) - strong, then $Im(\tau)$ is a subring of T_2. In this last case, $Im(\kappa) = (\alpha(A_1), \mu(M_1), \nu(N_1), \beta(B_1))$ is an MC and $Im(\tau)$ is the ring of the context $Im(\kappa)$.

Proof. The axiom under addition is trivial, while the axiom under multiplication is proved as follows.

$$\begin{bmatrix} a & n \\ m & b \end{bmatrix} \begin{bmatrix} a' & n' \\ m' & b' \end{bmatrix} = \begin{bmatrix} aa' + \langle n, m' \rangle_{A_1} & an' + nb' \\ ma' + bm' & \langle m, n' \rangle_{B_1} + bb' \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \alpha(a)\alpha(a') + \langle \nu(n), \mu(m') \rangle_{A_2} & \alpha(a)\nu(n') + \nu(n)\beta(b') \\ \mu(m)\alpha(a') + \beta(b)\mu(m') & \langle \mu(m), \nu(n') \rangle_{B_2} + \beta(b)\beta(b') \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(a) & \nu(n) \\ \mu(m) & \beta(b) \end{bmatrix} \begin{bmatrix} \alpha(a') & \nu(n') \\ \mu(m') & \beta(b') \end{bmatrix}$$

Remaining parts can be proved by using commutative diagrams given in the construction of the MC morphisms.

4. Applications

4.1. Projective Morita Contexts (PMC). An MC K is termed as a PMC, the abbreviation for a projective Morita context, if the two Morita context maps are surjective. K is a PMC iff it satisfies Morita Theorems I and II ([3, Section 3.12]). The term PMC is used in [7] just to shrink the phrase "Morita context satisfies Morita Theorems I and II". We also say that an MC ring T is a PMC ring if its context K is a PMC.

Theorem 4.1.1. Let $\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K \to K'$ be a context morphism between MCs $K = (A, M, N, B)$ and $K' = (A', M', N', B')$.

(i) If K' is a PMC, α and β are monomorphisms, and μ and ν are (β, α) and (α, β) - epimorphisms, respectively, then K is a PMC.

(ii) If K is a PMC and κ an epimorphism then K' is also a PMC.

Proof. (i) Let K' be a PMC, that is the two Morita context maps $\langle , \rangle_{A'}$, and $\langle , \rangle_{B'}$ are epimorphisms. Consider the commutative daigram:

$$\begin{array}{ccc} M \otimes_A N & \xrightarrow{\langle \rangle_B} & B \\ {\scriptstyle \mu \bar{\otimes} \nu} \downarrow & & \downarrow {\scriptstyle \beta} \\ M' \otimes_{A'} N' & \xrightarrow[\langle , \rangle_{B'}]{} & B' \end{array}$$

Since μ and ν are epic, $\mu \bar{\otimes} \nu$ is epic, also β is monic and $\langle , \rangle'_B$ is both monic and epic, so $\langle , \rangle_B$ is epic. Similarly $\langle , \rangle_A$ is also epic. Hence K is a PMC.

Proof of (ii) is similar to (i).

In this theorem in (ii) in fact we have proved that the homomorphic image of a PMC is a PMC. While in (i) we have proved its partial converse. The combined result is the following

Corollary 4.1.2. Let $K = (A, M, N, B)$ and $K' = (A, M', N', B)$ be two MCs with the common base rings A and B. If $\kappa = (1_A, \mu, \nu, 1_B) : K \to K'$ is an epimorphism, then K is a PMC.

4.2. Nondegenerate Morita Context

Recall that an MC $K = (A, M, N, B)$ is nondegenerate iff it satisfies any one of the conditions of following lemma. For the proof one may refer to [5,8, & 9]. Let us also an MC ring T nondegenerate if its context K is nondegenerate.

Lemma 4.2.1. For an MC $K = (A, M, N, B)$ the following are equivalent.

(i) M_A, N_B ${}_BM$ and ${}_AN$ are faithful and the two MC maps $\langle , \rangle_A$ and $\langle , \rangle_B$ are also faithful.

(ii) M_A is faithful and $\langle N, m \rangle_A \neq 0$ whenever $0 \neq m \in M$.

(iii) All A-modules and B-modules associated are I-free and J-free.

Theorem 4.2.2. Let $\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K \to K'$ be a homomorphism of MCs K and K' such that α and μ are monomorphisms and ν is an epimorphism. If the MC K' (respt. MC ring T) is nondegenerate, then K (respt. T) is also nondegenerate.

Proof. Assume that $M_A a = 0_M$, for some $a \in A$. Then for all $m \in M$, $ma = 0_M$. Or

$$\mu(ma) = \mu(m)\alpha(a) = 0_{M'}$$

But $M'_{A'}$ is faithful, so $\alpha(a) = 0$ and since α is a monomorphism, $a = 0_A$. Hence M_A is faithful.

Next, assume that $\langle N, m\rangle_A = 0_A$. Then

$$\alpha\langle N, m\rangle_A = \langle \nu(N), \mu(m)\rangle_{A'} = \langle N', \mu(m)\rangle_{A'} = 0_{A'}$$

which implies that $\mu(m) = 0$. But according to the hypothesis, μ is monic, $m = 0$. Hence both conditions of Lemma 4.2.1 (ii) are satisfied and which implies that K be nondegenerate.

Theorem 4.2.3. Let $\kappa = \langle \alpha, \mu, \nu, \beta\rangle : K \to \dot{K}'$ be a morphism of an *MC* K into another *MC* K' such that α and μ are isomorphisms. If K (respt. T) is nondegenerate, then K' (respt. T') is also nondegenerate.

Proof. Let the *MC* $K = (A, M, N, B)$ be nondegenerate. Assume that in $K' = (A', M', N', B')$, $M'a' = 0_{M'}$ for some $a' \in A'$. Since $\mu(M) \subseteq M'$ and α is an epimorphism, there exists $a \in A$ such that

$$M'a' = \mu(M)\alpha(a) = \mu(Ma) = \{0_{M'}\}$$

Since μ is monic, $Ma = \{0_M\}$ and as M_A is faithful, $a = 0_A$, which implies $a' = 0_B$.

Now assume that $\langle N', n'\rangle = \{0_{A'}\}$. Since $\nu(N) \subseteq N'$ and μ is epic, then for some $m \in M$

$$\langle \nu(N), \mu(m)\rangle_{A'} = \alpha\langle N, m\rangle = \{0_{A'}\}$$

But α is monic, so $\langle N, m\rangle = \{0_A\}$ which implies that $m = 0_M$. Hence $\mu(m) = m' = 0$, and by Lemma 4.2.1 we conclude that K' is nondegenerate.

4.3. Context Existence/Ring Extensions

This section poses another example of morphisms between Morita contexts. In fact, in the following context extensions and ring extensions are mutually studied.

Let A and B be rings and as previously, $\alpha : A \to B$, a ring homomorphism such that $\alpha(I_A) = I_B$. Assume that M is an A - module and $D = \text{End}_A(M)$, the ring of endomorphisms on M_A. Next we assume that $E = \text{End}_B(M \otimes_A B)$, the ring of endomorphisms on $M \otimes_A B$ in $Mod - B$. Then $M \otimes_A B$ becomes an (E, B) - bimodule, and there is a ring homomorphism $\sigma : D \to E$ defined by

$$\sigma(d)(m \otimes b) = d(m) \otimes b,$$

where $b \in B,\ d \in D$ and $m \in M$. Clearly, $\sigma(I_D) = I_E$.

The Context Induced from the Derived Contexts. Now consider the dual module $M^* = \text{Hom}_A(M, A)$ of M. Let $K = (A, M, M^*, D)$ be the derived context of M. Instead of putting some conditions on M, assume that $M^* \otimes_D E$ is left B-module. We will continue this assumption up to the end. Now we claim that $K' = (B, M \otimes_A B,\ M^* \otimes_D E, E)$ is a Morita context. We call it a *context induced from the derived context of* M. Indeed

$$\begin{aligned}(M^* \otimes_D E) \otimes_E (M \otimes_A B) &\cong M^* \otimes_D M \otimes_A B \\ &\longrightarrow A \otimes_A B \\ &\cong B\end{aligned}$$

where the arrow is the MC map $\langle , \rangle_A : M^* \otimes_D M \to A$ of the first MC K. Simiarly

$$\begin{aligned}(M^* \otimes_A B) \otimes_B (M^* \otimes_D E) &\cong M \otimes_A M^* \otimes_D E \\ &\longrightarrow D \otimes_D E \\ &\cong E\end{aligned}$$

The Morphism Between Derived and Induced Contexts. Assume that $\kappa = \langle \alpha, \mu, \nu, \sigma \rangle : K \to K'$, is a map in which $\alpha : A \to B$ and $\sigma : D \to E$ are as given above, $\mu : M \to M \otimes_A B$ is defined by $\mu(m) = m \otimes 1_B$ for all $m \in M$ and $\nu : M^* \to M^* \otimes_D B$ is defined by $\nu(m^*) = m^* \otimes 1_E$. Then we have

Theorem 4.3.1. If $A, B, D, E, M, M^*, \alpha, \sigma, \mu$ and ν are as given above, then $\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K \to K'$ is an MC morphism.

Proof. First we verify that μ and ν are (σ, α) - and (α, σ) - homomorphisms, respectively. Indeed, for all $a \in A$, $d \in D$, $m \in M$, and $m^* \in M^*$, we can write the following relations

$$\begin{aligned}\mu(dma) &= \sigma(d)(m \otimes 1_B)\alpha(a) \\ &= d(m) \otimes \alpha(a)\end{aligned}$$

and

$$\begin{aligned}\nu(am^*d) &= \alpha(a)(m^* \otimes 1_E)\sigma(d) \\ &= \alpha(a)[m^* \otimes \sigma(d)]\end{aligned}$$

Next we establish the commutativity of the following diagrams

$$\begin{array}{ccc} M \otimes_A M^* & \xrightarrow{\langle , \rangle_B} & D \\ {\scriptstyle \mu \bar{\otimes} \nu}\downarrow & & \downarrow{\scriptstyle \sigma} \\ (M \otimes_A B) \otimes_B (M^* \otimes_D E) & \xrightarrow{\langle , \rangle_E} & E \end{array}$$

and

$$\begin{array}{ccc} M^* \otimes_D M & \xrightarrow{\langle,\rangle_A} & A \\ \nu\bar{\otimes}\mu \big\downarrow & & \big\downarrow \alpha \\ (M^* \otimes_D E) \otimes_E (M \otimes_A B) & \xrightarrow{\langle,\rangle_B} & B \end{array}$$

In the first diagram, in one direction

$$[\sigma \circ \langle,\rangle_D] \sum (m_i \otimes m_i^*) = \sigma[\sum \langle m_i, m_i^* \rangle_D \in E$$

and from the other direction we get

$$\begin{aligned} [\langle,\rangle_E \circ \mu\bar{\otimes}\nu] \sum (m_i \otimes m_i^*) &= \langle,\rangle_E \sum [\mu(m_i) \otimes \nu(m_i^*)] \\ &= \sum \langle m_i \otimes 1_B, m_i^* \otimes 1_E \rangle_E \\ &\in E \end{aligned}$$

Note that, for any $n \in M$ and $b \in B$

$$\begin{aligned} \sigma\langle m, m^* \rangle_D (n \otimes b) &= \langle m, m^* \rangle_D n \otimes b \\ &= m[m^*(n)] \otimes b \end{aligned}$$

Similarly

$$\begin{aligned} (m \otimes 1_B) \otimes (m^* \otimes 1_E) &\longrightarrow (m \otimes m^*) \otimes 1_E \\ &\longrightarrow \langle m, m^* \rangle_D \otimes 1_E \\ &\longrightarrow \langle m, m^* \rangle_D 1_E \in E \end{aligned}$$

Then, by evaluating $n \otimes b$ at the last function, we get

$$\langle m, m^* \rangle_D 1_E (n \otimes b) = m[m^*(n)] \otimes b$$

Hence we conclude

$$\langle,\rangle_E \circ \mu\bar{\otimes}\nu = \sigma \circ \langle,\rangle_D$$

For the second diagram one can similarly prove that

$$[\alpha \circ \langle,\rangle_A] = [\langle,\rangle_B \circ \nu\bar{\otimes}\mu]$$

Hence we conclude that κ is morphism between contexts.

The following is an immediate consequence of above theorem.

Corollary 4.3.2. Let T and T' be the rings of MCs K and K', respectively. Then the MC map $\kappa : K \to K'$ of above theorem induces the ring homomorphism $\tau : T \to T'$.

4.4. Static Modules

M-Static Modules. An object V of Mod $-A$ is static if it remains invariant under the composition of the adjoint functors $\mathrm{Hom}_A(M,-)$ and $-\otimes_D M$. In particular, the ring A as an A - module is M - static if $M^* \otimes_D M \cong A$ via the natural isomorphism $m^* \otimes m \to m^*(m)$ for all $m \in M$ and $m^* \in M^*$.

In case the ring A is M – static, by [6, Lemma 3.5] we have

Lemma 4.4.1. If the ring A is M – static, then

$$M^* \otimes_D E \cong (M \otimes_A B)^*$$

as E-modules via the map

$$(m^* \otimes f)\left(\sum_{i=I}^{k} m_i \otimes b_i\right) \mapsto \sum_{j=1}^{l} \langle m^*, n_j\rangle c_j$$

where $m_i, n_j \in M$, $m^* \in M^*$ and $b_i, c_j \in B$ and $f \in E$ is such that

$$f\left(\sum_{i=I}^{k} m_i \otimes b_i\right) = \sum_{j=1}^{l} n_j \otimes c_j$$

Hence we state that

Theorem 4.4.2. If the ring A is M – static, then the induced derived contex of M is isomorphic to the derived context of $M \otimes_A B$. The respective rings of contexts are also isomorphic.

Proof. It follows from Theorem 4.3.1 and Lemma 4.4.1 that there is an MC morphism from the induced derived context of M to the derived context of $M \otimes_A B$ given by

$$\kappa' = \langle \alpha', \mu', \nu', \beta' \rangle : K' \to K''$$

where

$$K'' = \{B, M \otimes_A B, (M \otimes_A B)^*, E\}$$

Clearly, α', β' and μ' are the identical maps while

$$\nu' : M^* \otimes_D E \longrightarrow (M \otimes_A B)^*$$

is an isomorphism as given in the Lemma 4.4.1. Hence $\kappa' : K' \to K''$ is an MC isomorphism. The last statement follows from Corollary 4.3.2.

Corollary 4.4.3. If the ring A is M – static, then there always is a morphism (respt. ring homomorphism) between the derived contexts (respt. rings of derived contexts) of M and of $M \otimes_A B$.

Proof. By Proposition 3.1.3, the composition of the MC morphisms

$$K \xrightarrow{\kappa} K' \xrightarrow{\kappa'} K''$$

is an MC morphism.

If the derived context of M is a PMC, then A becomes $M -$ static. By using Theorems 3.3 and 3.4 of [6] we restate that

Corollary 4.4.4. *(a)* If K, the derived context of M, is a PMC, then K', the induced derived context of M, and the derived context K'' of $M \otimes_A B$ are also PMCs.
(b) If $\alpha : A \to B$ is a monomorphism then K is a PMC if and only if K' (or K'') is a PMC.

4.5. Purity

Let the ring homomorphism $\alpha : A \to B$ be a pure homomorphism. Then for every $M \in$ Mod $- A$, the α-homomorphism $\mu : M \otimes_A B$ is injective (Example 2.1.2).

Recently, in studying relationship between effective descent morphisms and pure homomorphisms, Mesablishvili in [4;3.2. Theorem] proved that

Theorem 4.5.1. If $\alpha : A \to B$ is a pure homomorphism of commutative rings and if for any $M \in$ Mod $- A$, $M \otimes_A B$ is f.g., flat, and f.g. flat, and f.g. projective in Mod $- B$, then M is f.g., flat, f.g. flat, and f.g. projective in Mod $- A$, respectively.

By using Corollary 4.4.4 (b), we can add one more property in the above list without involving commutativity of rings.

Corollary 4.5.2. If $\alpha : A \to B$ is a pure (or simply injective), then M is a progenerator of Mod $- A$ if and only if $M \otimes_A B$ is a progenerator of Mod $- B$.

Proof. Recall that M is a progenerator of Mod $- A$ if and only if any arbitrary MC $K = (A, M, N, C)$ is a PMC (cf. [3 & 7]). Then ${}_AN_C \cong M^*$ and $C \cong$ End $(M_A) = D$. This holds if and only if the derived context of $M, K = (A, M, M^*, D)$ is a PMC. Note that, if $\alpha : A \to B$ is a pure then it is also injective. By Corollary 4.4.4(b), K is a PMC if and only if the induced context K' of $M \otimes_A B$ is a PMC, which holds if and only if $M \otimes_A B$ is a progenerator of Mod $- B$.

References

[1] Amitsur, S.A. : *Rings of quotients and Morita contexts*, J. Algebra, 17 (1971) 273-298.

[2] Hungerford, T.W. : *Algebra*, Springer-Verlag, (1974).

[3] Jacobson, N. : *Basic Algebra-II*, Freeman & Co., (1980).

[4] Mesablishvili, B. : *On some properties of pure morphisms of commutative rings*, Th. & App. of Categories, 10 (2002) 180-186.

[5] Müller, B.J. : *The quotient categories of a Morita context*, J. Algebra, 28 (1974) 389-407.

[6] Nauman, S.K. : *Static modules, stable Clifford theory, and Morita similarities of rings*, J. Algebra, 143 (1991) 498-504.

[7] Nauman, S.K. : *Static modules, stable Clifford theory, and colocalization - localization*, J. Algebra, 170 (1994) 400-421.

[8] Zhengping, Z. : *Endomorphism rings of nondegenerate modules*, Proc. AMS 120 (1994) 85-88.

[9] Zhengping, Z. : *Correspondence theorems for non-degenerate modules and their endomorphism rings*, Proc. AMS 121 (1994) 25-32.